

ON CALCULATIONS OF THE TWISTED ALEXANDER IDEALS FOR SPATIAL GRAPHS, HANDLEBODY-KNOTS AND SURFACE-LINKS

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ABSTRACT. There are many studies about twisted Alexander invariants for knots and links, but calculations of twisted Alexander invariants for spatial graphs, handlebody-knots, and surface-links have not been demonstrated well. In this paper, we give some remarks to calculate the twisted Alexander ideals for spatial graphs, handlebody-knots and surface-links, and observe their behaviors. For spatial graphs, we calculate the invariants of Suzuki's theta-curves and show that the invariants are nontrivial for Suzuki's theta-curves whose Alexander ideals are trivial. For handlebody-knots, we give a remark on abelianizations and calculate the invariant of the handlebody-knots up to six crossings. For surface-links, we correct Yoshikawa's table and calculate the invariants of the surface-links in the table.

1. INTRODUCTION

The Alexander ideal is a knot invariant derived from the fundamental group of the exterior of a knot with an abelianization, which can be specified by the meridian of the knot. The twisted Alexander ideal is a generalization of the Alexander ideal, which is derived from the fundamental group, an abelianization and a group representation. There are two versions of the twisted Alexander invariant introduced by Lin [12] and Wada [18]. In this paper, we follow Wada's version. The Alexander ideal can be defined not only for a knot but also for a finitely presentable group with an abelianization. The twisted Alexander ideal can also be defined for a finitely presentable group with an abelianization and a group representation. Since calculations of twisted Alexander ideals have not been demonstrated well except for knots, we give some remarks to calculate the twisted Alexander ideals for spatial graphs, handlebody-knots and surface-links, and observe their behaviors.

For spatial graphs, we focus on Suzuki's theta- n curve Θ_n as illustrated in Fig. 3.1 where n is a positive integer satisfying $n \geq 3$. Alexander ideals for Θ_n were calculated in [15, 17] and as we will demonstrate later, it is trivial if $n \equiv 1, 5 \pmod{6}$. In this case, we define a family of group representations and give a formula of the twisted Alexander ideals for Θ_n . It follows from the formula that Suzuki's theta- n curve is nontrivial if $n \equiv 1, 5 \pmod{6}$, although it can not be shown by the Alexander ideal.

2010 *Mathematics Subject Classification.* Primary 57M05; Secondary 57M15, 57Q45.

Key words and phrases. Spatial graph, Handlebody-knot, Surface-link, twisted Alexander ideal.

The first author was supported by JSPS KAKENHI Grant Number 24740037.

The second author was supported by JSPS KAKENHI Grant Number 24540094.

The third author was supported by JSPS KAKENHI Grant Number 25800052.

For handlebody-knots, we focus on the handlebody-knots in the table [8] of genus two handlebody-knots up to six crossings. Since a meridian system of a handlebody-knot is not unique, a group representation and an abelianization can not be specified via meridian systems. Then we sum up the twisted Alexander ideals over representations and abelianizations, and obtain an invariant of matrix form. We confirm that the twisted Alexander ideal works better than the number of representations as expected.

For surface-links, we focus on the surface-links in Yoshikawa's table [19], where each surface-link is represented by a ch-diagram. We correct three calculations in his table and calculate the twisted Alexander ideals for surface-links in the table. We also remark that there exists a pair of two surface-links such that their twisted Alexander ideals differ but that their Alexander ideals coincide.

This paper is organized as follows. In Section 2, we review the twisted Alexander ideal for a finitely presentable group G associated with an epimorphism from G to an abelian group and a group representation from G to a matrix group. In Section 3, we give a formula of the twisted Alexander ideal for Suzuki's theta-curves. In Section 4, we introduce an invariant of matrix form and give a table of the invariants for genus two handlebody-knots up to six crossings. In Section 5, we calculate the twisted Alexander ideal for surface-links in Yoshikawa's table and correct three calculations in his table. In this paper, we denote by \mathbb{Z}_p the cyclic group of order p , namely $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$.

2. TWISTED ALEXANDER IDEALS

In this section, we give a brief review of the twisted Alexander ideals for finitely presentable groups. Let P be a commutative ring with unity 1. For two matrices A, B over P , we say that A and B are *elementarily equivalent* if they are transformed into each other by a finite sequence of the *elementary operations*:

- (1) Permuting rows and columns,
- (2) Adjoining a row of zeros; $M \rightarrow \begin{pmatrix} M \\ \mathbf{0} \end{pmatrix}$,
- (3) Adding to some row a linear combination of other rows,
- (4) Adding to some column a linear combination of other columns,
- (5) Adding a new row and a new column such that the entry in the intersection of the new row and new column is 1, and the all of the remaining entries in the new column are zero; $M \rightarrow \begin{pmatrix} M & \mathbf{0} \\ * & 1 \end{pmatrix}$,
- (6) The inverses of (1), (2), (3), (4) and (5).

Then we denote by $A \sim B$. Note that the multiplication of a row and column by a unit of P can be realized by elementary operations. Let A be a $\infty \times s$ matrix over P such that only the first t rows contain nonzero entries. Then the d -th *elementary ideal* $E_d(A)$ of A is defined by the ideal of P generated by all $(s-d)$ -minors of A if $0 < s-d \leq t$, (0) if $s-d > t$ and (1) = P if $s-d \leq 0$. It is known that elementarily equivalent two matrices over P have the same sequence of elementary ideals [3].

For a group G and a ring R , we denote the group ring of G over R by RG . Let F_s be the free group with rank s generated by x_1, x_2, \dots, x_s . The Fox's free derivative [4] with respect to x_i is an additive map

$$\frac{\partial}{\partial x_i} : \mathbb{Z}F_s \rightarrow \mathbb{Z}F_s$$

which satisfies

$$\frac{\partial x_j}{\partial x_i} = \begin{cases} e & (i = j) \\ 0 & (i \neq j) \end{cases}, \quad \frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i}$$

for $u, v \in F_s$. In particular, the map satisfies

$$\frac{\partial x_i^p}{\partial x_i} = \begin{cases} e + x_i + x_i^2 + \cdots + x_i^{p-1} & (p > 0) \\ 0 & (p = 0) \\ -x_i^{-1} - x_i^{-2} - \cdots - x_i^p & (p < 0). \end{cases}$$

For $r_1, r_2, \dots, r_t \in F_s$, let

$$G = \langle x_1, x_2, \dots, x_s \mid r_1, r_2, \dots, r_t \rangle$$

be a finitely presented group with s generators and t relators. Let $\phi : F_s \rightarrow G$ be the canonical epimorphism and α an epimorphism from G to an abelian group

$$G_0 = \langle t_1, \dots, t_r \mid t_1^{k_1}, \dots, t_r^{k_r}, [t_i, t_j] \ (1 \leq i < j \leq r) \rangle$$

for some non-negative integers k_1, \dots, k_r , where $[t_i, t_j]$ denotes the commutator of t_i and t_j . Then for a ring R , the group ring RG_0 may be identified with the quotient ring of Laurent polynomial ring $R[t_1^{\pm 1}, \dots, t_r^{\pm 1}]/(t_1^{k_1} - 1, \dots, t_r^{k_r} - 1)$. We denote by $\tilde{\phi}$ and $\tilde{\alpha}$ the linear extensions of ϕ and α from $\mathbb{Z}F_s$ to $\mathbb{Z}G$ and from $\mathbb{Z}G$ to $\mathbb{Z}G_0$, respectively. Then we call the $\infty \times s$ matrix

$$A(G, \tilde{\alpha}) = \begin{pmatrix} \tilde{\alpha} \circ \tilde{\phi} \left(\frac{\partial r_i}{\partial x_j} \right) \\ O \end{pmatrix}$$

each of whose entries belongs to $\mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]/(t_1^{k_1} - 1, \dots, t_r^{k_r} - 1)$ the *Alexander matrix of G associated with α* , where O stands for $\infty \times s$ zero matrix. We call the d -th elementary ideal $E_d(A(G, \tilde{\alpha}))$ of $A(G, \tilde{\alpha})$ the *d -th Alexander ideal of G associated with α* .

Moreover, let $GL(n; R)$ be the general linear group of degree n over a ring R and $\rho : G \rightarrow GL(n; R)$ a group representation. We denote by $\tilde{\rho}$ the linear extension of ρ from $\mathbb{Z}G$ to the matrix ring $M_n(R) = M(n, n; R)$. Then the tensor product homomorphism

$$\tilde{\rho} \otimes \tilde{\alpha} : \mathbb{Z}G \rightarrow M_n(R[t_1^{\pm 1}, \dots, t_r^{\pm 1}]/(t_1^{k_1} - 1, \dots, t_r^{k_r} - 1))$$

is defined by

$$(\tilde{\rho} \otimes \tilde{\alpha}) \left(\sum r_i g_i \right) = \sum r_i \alpha(g_i) \rho(g_i) \quad (r_i \in \mathbb{Z}, g_i \in G).$$

Then we call the $\infty \times s$ matrix

$$A(G, \tilde{\rho} \otimes \tilde{\alpha}) = \begin{pmatrix} (\tilde{\rho} \otimes \tilde{\alpha}) \circ \tilde{\phi} \left(\frac{\partial r_i}{\partial x_j} \right) \\ O \end{pmatrix}$$

each of whose entries belongs to $M_n(R[t_1^{\pm 1}, \dots, t_r^{\pm 1}]/(t_1^{k_1} - 1, \dots, t_r^{k_r} - 1))$ the *twisted Alexander matrix of G associated with α and ρ* , where O stands for $\infty \times s$ matrix consisting entirely of $n \times n$ zero matrix. We regard $A(G, \tilde{\rho} \otimes \tilde{\alpha})$ as $\infty \times ns$ matrix in $M(nt, ns; R[t_1^{\pm 1}, \dots, t_r^{\pm 1}]/(t_1^{k_1} - 1, \dots, t_r^{k_r} - 1))$ where only the first nt rows contain nonzero entries. Then we call the d -th elementary ideal $E_d(A(G, \tilde{\rho} \otimes \tilde{\alpha}))$ of $A(G, \tilde{\rho} \otimes \tilde{\alpha})$ the *d -th twisted Alexander ideal of G associated with ρ and α* .

Since any two presentations of G are related by Tietze transformations and these transformations induce a sequence of elementary operations, it follows that the Alexander matrix $A(G, \tilde{\alpha})$ of G associated with α does not depend on the choice of a presentation of G . Namely we have the following.

Theorem 2.1. *The sequence of Alexander ideals of G associated with α does not depend on the choice of a presentation of G .* \square

In a similar way, the d -th twisted Alexander matrix of G associated with ρ and α does not depend on the choice of a presentation of G . In addition, let $\rho' : G \rightarrow GL(n; R)$ be a group representation which is conjugate to ρ , that is, there exists $B \in GL(n; R)$ such that $\rho'(x) = B\rho(x)B^{-1}$ for any $x \in G$. Then it is not hard to see that the twisted Alexander matrix $A(G, \tilde{\rho}' \otimes \tilde{\alpha})$ of G associated with α and ρ' is elementarily equivalent to the twisted Alexander matrix $A(G, \tilde{\rho} \otimes \tilde{\alpha})$ of G associated with α and ρ . Therefore we have the following.

Theorem 2.2. *The sequence of twisted Alexander ideals of G associated with α and ρ does not depend on the choice of a presentation of G and a representative element in the conjugacy class of ρ .* \square

In particular for $t = s - 1$ and $G_0 = \langle t_1 \mid \emptyset \rangle$, it is known that the first Alexander ideal $E_1(A(G, \tilde{\alpha}))$ is always principal and its generator is called the *Alexander polynomial* of G associated with α [3]. Moreover, a specific generator of $E_n(A(G, \tilde{\rho} \otimes \tilde{\alpha}))$ produces the *twisted Alexander polynomial* of G associated with α and ρ . We refer the reader to [18, 12, 7] for the precise definition of the twisted Alexander polynomial.

3. SPATIAL GRAPHS

Let Γ be a finite and labeled graph embedded in the 3-sphere S^3 . Then Γ is called a *spatial graph*. Two spatial graphs are said to be *equivalent* if there exists an orientation-preserving self-homeomorphism on S^3 which sends one to the other as labeled graphs. A spatial graph Γ is said to be *trivial* if there exists a spatial graph Γ' contained in a 2-sphere in S^3 such that Γ is equivalent to Γ' . For a spatial graph Γ , the fundamental group of the spatial graph complement $G_\Gamma = \pi_1(S^3 - \Gamma)$ is finitely presentable and we can obtain a group presentation of deficiency $1 - \beta_0(\Gamma) + \beta_1(\Gamma)$, where $\beta_i(\Gamma)$ denotes the i -th Betti number of Γ [11]. In particular, if Γ is trivial, then G_Γ is isomorphic to the free group of rank $\beta_1(\Gamma)$. Let l be a 1-dimensional cycle (in the sense of homology) with integral coefficients on Γ . Then we define a homomorphism α_l from G_Γ to $\langle t \mid \emptyset \rangle$ by $\alpha_l(g) = t^{\text{lk}(g, l)}$ for any element g in G_Γ , where lk denotes the linking number in S^3 . Then the collection of the Alexander ideals of G_Γ associated with α_l is an invariant of Γ . Moreover, let ρ be a group representation from G_Γ to $SL(n; R)$. Then the collection of the twisted Alexander ideals of G_Γ associated with α_l and all possible ρ is also an invariant of Γ . In particular the following holds by the definition of the elementary ideals.

Lemma 3.1. *If Γ is trivial, then it follows that*

$$(3.1) \quad E_d(A(G_\Gamma, \tilde{\alpha}_l)) = \begin{cases} (0) & (d < \beta_1(\Gamma)) \\ (1) & (d \geq \beta_1(\Gamma)) \end{cases},$$

$$(3.2) \quad E_d(A(G_\Gamma, \tilde{\rho} \otimes \tilde{\alpha}_l)) = \begin{cases} (0) & (d < n\beta_1(\Gamma)) \\ (1) & (d \geq n\beta_1(\Gamma)) \end{cases}$$

for any l and ρ . \square

For a positive integer $n \geq 3$, let Θ_n be *Suzuki's theta- n curve* [17] which is a spatial graph represented by the diagram illustrated in Fig. 3.1. Note that Θ_3 is also called *Kinoshita's theta curve*. We denote G_{Θ_n} by G_n simply.

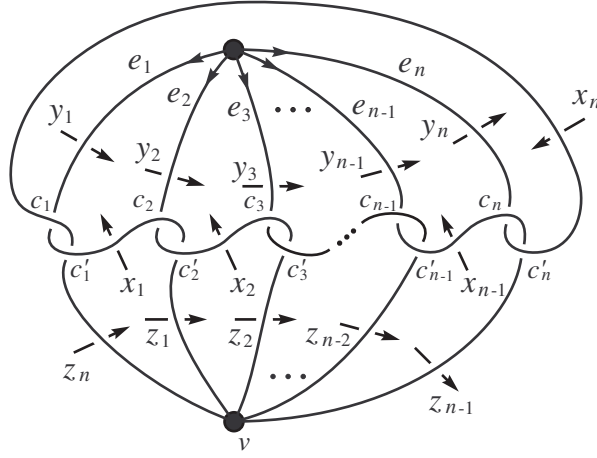


FIGURE 3.1. Suzuki's theta- n curve

It can be easily seen that any proper spatial subgraph of Θ_n is trivial. The non-triviality of Θ_n for all n was shown in [16] first by a geometric argument and also was shown in [13] by an application of a symmetry of Θ_n and the branched cover of S^3 . Our purpose in this section is to demonstrate an effectiveness of (twisted) Alexander ideals by showing the non-triviality of Θ_n as an application of it.

Lemma 3.2. ([17]) G_n has a presentation

$$\langle x_1, x_2, \dots, x_n \mid (x_1 x_n x_1^{-1}) (x_2 x_1 x_2^{-1}) \cdots (x_n x_{n-1} x_n^{-1}) \rangle.$$

Proof. By taking a Wirtinger presentation on the diagram in Fig. 3.1, we have

$$G_n \cong \langle x_i, y_i, z_i \ (i = 1, 2, \dots, n) \mid r_i, r'_i, r'' \ (i = 1, 2, \dots, n) \rangle,$$

where r_i , r'_i and r'' are relators corresponding to the crossings c_i , c'_i and vertex v as illustrated in Fig. 3.1, respectively. Moreover we have

$$\begin{aligned} r_i &= x_i x_{i-1} y_i^{-1} x_{i-1}^{-1} \ (i = 1, 2, \dots, n), \\ r'_i &= z_{i-1} x_i x_{i-1}^{-1} x_i^{-1} \ (i = 1, 2, \dots, n), \\ r'' &= z_n z_1 z_2 \cdots z_{n-1}, \end{aligned}$$

where suffix i is taken modulo n . Then by deleting generators y_i, z_i and relators r_i, r'_i by Tietze transformations, we have the result. \square

We denote the 1-dimensional cycle $e_i - e_n$ on Θ_n by l_i for $i = 1, 2, \dots, n-1$. Note that $\{l_1, l_2, \dots, l_{n-1}\}$ is a basis of $H_1(\Gamma; \mathbb{Z})$, namely $\beta_1(\Gamma) = n-1$. Then

$l = \sum_{i=1}^{n-1} l_i$ is also a 1-dimensional cycle on Θ_n . We denote the homomorphism α_l by α_n . For generators x_1, x_2, \dots, x_n of G_n , we have

$$\begin{aligned}\alpha_n(x_i) &= t^{\text{lk}(x_i, l)} = t^{\text{lk}(x_i, l_i)} = t \quad (i = 1, 2, \dots, n-1), \\ \alpha_n(x_n) &= t^{\text{lk}(x_n, l)} = t^{\sum_{i=1}^{n-1} \text{lk}(x_n, l_i)} = t^{1-n}.\end{aligned}$$

Then Sato calculated the Alexander ideal of $A(G_n, \tilde{\alpha}_n)$ as follows.

Theorem 3.3. (Sato [15])

$$E_d(A(G_n, \tilde{\alpha}_n)) = \begin{cases} (0) & (d < n-1) \\ (1-t+t^2, 1-t^n) & (d = n-1) \\ (1) & (d \geq n) \end{cases}.$$

In particular for $d = n-1$, it follows that

$$E_{n-1}(A(G_n, \tilde{\alpha}_n)) = \begin{cases} (1) & (n \equiv 1, 5 \pmod{6}) \\ (3, 1+t) & (n \equiv 2, 4 \pmod{6}) \\ (2, 1-t+t^2) & (n \equiv 3 \pmod{6}) \\ (1-t+t^2) & (n \equiv 0 \pmod{6}) \end{cases}.$$

In [15], Sato calculated Alexander ideals for a wide class of spatial graphs containing Θ_n . In the following, we describe a proof of Theorem 3.3 for reader's convenience.

Proof. By Lemma 3.2, it is clear that $E_d(A(G_n, \tilde{\alpha}_n)) = (0)$ if $d < n-1$ and (1) if $d \geq n$. So we calculate $E_{n-1}(A(G_n, \tilde{\alpha}_n))$. We put

$$r = (x_1 x_n x_1^{-1}) (x_2 x_1 x_2^{-1}) \cdots (x_n x_{n-1} x_n^{-1}).$$

Then by a direct calculation we have

$$(3.3) \quad \frac{\partial r}{\partial x_i} = \left\{ \prod_{m=1}^{i-1} (x_m x_{m-1} x_m^{-1}) \right\} \{ e - (x_i x_{i-1} x_i^{-1}) + (x_i x_{i-1} x_i^{-1}) x_{i+1} \}$$

for $i = 1, 2, \dots, n-1$ and

$$(3.4) \quad \frac{\partial r}{\partial x_n} = x_1 + \prod_{m=1}^{n-1} (x_m x_{m-1} x_m^{-1}) - r.$$

Thus we have

$$(3.5) \quad \tilde{\alpha}_n \circ \tilde{\phi} \left(\frac{\partial r}{\partial x_i} \right) = t^{i-1-n} (1-t+t^2) \quad (i \neq 1, n-1),$$

$$(3.6) \quad \tilde{\alpha}_n \circ \tilde{\phi} \left(\frac{\partial r}{\partial x_1} \right) = t^{1-n} (t^{n-1} - 1 + t),$$

$$(3.7) \quad \tilde{\alpha}_n \circ \tilde{\phi} \left(\frac{\partial r}{\partial x_{n-1}} \right) = t^{-n} (t^{n-2} - t^{n-1} + 1).$$

Note that

$$(3.8) \quad 1 - t^n = (1 - t + t^2) - t(t^{n-1} - 1 + t),$$

$$(3.9) \quad t^{n-2} - t^{n-1} + 1 = t^{-1} (1 - t^n) + t^{-1} (t^{n-1} - 1 + t).$$

Then by (3.5),(3.6),(3.7),(3.8) and (3.9), it follows that

$$\begin{aligned}
E_{n-1}(A(G_n, \tilde{\alpha}_n)) &= (t^{n-1} - 1 + t, 1 - t + t^2, t^{n-2} - t^{n-1} + 1) \\
&= (1 - t^n, t^{n-1} - 1 + t, 1 - t + t^2, t^{n-2} - t^{n-1} + 1) \\
&= (1 - t^n, t^{n-1} - 1 + t, 1 - t + t^2) \\
&= (1 - t^n, 1 - t + t^2).
\end{aligned}$$

Namely we obtain the first half of the theorem. Next we show the second half. Note that the remainder of dividing $1 - t^n$ by $1 - t + t^2$ equals 0 if $n \equiv 0 \pmod{6}$, $1 - t$ if $n \equiv 1 \pmod{6}$, $2 - t$ if $n \equiv 2 \pmod{6}$, 2 if $n \equiv 3 \pmod{6}$, $1 + t$ if $n \equiv 4 \pmod{6}$ and t if $n \equiv 5 \pmod{6}$. Therefore we have

$$(1 - t + t^2, 1 - t^n) = \begin{cases} (1 - t + t^2) & (n \equiv 0 \pmod{6}) \\ (1 - t + t^2, 1 - t) = (1) & (n \equiv 1 \pmod{6}) \\ (1 - t + t^2, 2 - t) = (3, 1 + t) & (n \equiv 2 \pmod{6}) \\ (1 - t + t^2, 2) & (n \equiv 3 \pmod{6}) \\ (1 - t + t^2, 1 + t) = (3, 1 + t) & (n \equiv 4 \pmod{6}) \\ (1 - t + t^2, t) = (1) & (n \equiv 5 \pmod{6}) \end{cases}.$$

This completes the proof. \square

Remark 3.4. Let α'_n be the homomorphism from G_n to $\langle t | t^n \rangle$ defined by $\alpha'_n(x_i) = t$ ($i = 1, 2, \dots, n$). Note that $\alpha'_n = \xi_n \circ \alpha_n$, where ξ_n is the canonical homomorphism from $\langle t | \emptyset \rangle$ to $\langle t | t^n \rangle$. Then Theorem 3.3 leads to the following result of Suzuki [17]:

$$E_d(A(G_n, \tilde{\alpha}'_n)) = \begin{cases} (0) & (d < n - 1) \\ (1 - t + t^2) & (d = n - 1) \\ (1) & (d \geq n) \end{cases}.$$

Note that if $n \equiv 1, 5 \pmod{6}$ then $1 - t + t^2$ is invertible in $\mathbb{Z}[t, t^{-1}]$ and therefore $E_d(A(G_n, \tilde{\alpha}'_n)) = (1)$ (this was pointed out in [14] first).

By Lemma 3.1 and Theorem 3.3, it follows that Θ_n is nontrivial for $n \equiv 0, 2, 3, 4 \pmod{6}$. In the case of $n \equiv 1, 5 \pmod{6}$, Theorem 3.3 does not work for showing the nontriviality of Θ_n . In the following, we apply twisted Alexander ideals to Θ_n in the case of $n \equiv 1, 5 \pmod{6}$.

Lemma 3.5. *Let n be a positive integer satisfying $n \geq 3$ and $n \equiv 1, 5 \pmod{6}$. Let ρ be a map from G_n to $SL(2; \mathbb{Z}_2)$ defined by*

$$\begin{aligned}
\rho(x_i) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad (i = 1, 2, \dots, 6k + 3), \\
\rho(x_{6k+4}) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(x_{6k+5}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\end{aligned}$$

if $n = 6k + 5$ ($k \geq 0$), and

$$\begin{aligned}
\rho(x_i) &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad (i = 1, 2, \dots, 6k + 3), \\
\rho(x_{6k+4}) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(x_{6k+5}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\
\rho(x_{6k+6}) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(x_{6k+7}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\end{aligned}$$

if $n = 6k + 7$ ($k \geq 0$). Then ρ is a group representation.

Proof. By Lemma 3.2, it is sufficient to show that $\rho(r)$ equals to the identity matrix E . In the case of $n = 6k + 5$ ($k \geq 0$), we have

$$(3.10) \quad \rho(x_1 x_{6k+5} x_1^{-1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(3.11) \quad \rho(x_{i+1} x_i x_{i+1}^{-1}) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad (i = 1, 2, \dots, 6k+2),$$

$$(3.12) \quad \rho(x_{6k+4} x_{6k+3} x_{6k+4}^{-1}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(3.13) \quad \rho(x_{6k+5} x_{6k+4} x_{6k+5}^{-1}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = E$$

in $SL(2; \mathbb{Z}_2)$. Then by (3.10), (3.11), (3.12) and (3.13), we have

$$\begin{aligned} \rho(r) &= \rho(x_1 x_{6k+5} x_1^{-1}) \rho(x_2 x_1 x_2^{-1}) \cdots \rho(x_{6k+5} x_{6k+4} x_{6k+5}^{-1}) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= E. \end{aligned}$$

In the case of $n = 6k + 7$ ($k \geq 0$), we have

$$(3.14) \quad \rho(x_1 x_{6k+7} x_1^{-1}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(3.15) \quad \rho(x_{i+1} x_i x_{i+1}^{-1}) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad (i = 1, 2, \dots, 6k+2),$$

$$(3.16) \quad \rho(x_{6k+4} x_{6k+3} x_{6k+4}^{-1}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$(3.17) \quad \rho(x_{6k+j} x_{6k+j-1} x_{6k+j}^{-1}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (j = 5, 6, 7).$$

Then by (3.14), (3.15), (3.16) and (3.17), we have

$$\begin{aligned} \rho(r) &= \rho(x_1 x_{6k+7} x_1^{-1}) \rho(x_2 x_1 x_2^{-1}) \cdots \rho(x_{6k+7} x_{6k+6} x_{6k+7}^{-1}) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 \\ &= E. \end{aligned}$$

Thus we have the desired conclusion. \square

Now we state our main theorem in this section.

Theorem 3.6. *Let n be a positive integer satisfying $n \geq 3$ and $n \equiv 1, 5 \pmod{6}$. Let ρ be a group presentation from G_n to $SL(2; \mathbb{Z}_2)$ defined in Lemma 3.5. Then it follows that*

$$E_d(A(G_n, \tilde{\rho} \otimes \tilde{\alpha}_n)) = \begin{cases} (0) & (d < 2n-2) \\ (1+t) & (d = 2n-2) \\ (1) & (d \geq 2n-1) \end{cases}.$$

Proof. We denote the composition map $(\tilde{\rho} \otimes \tilde{\alpha}_n) \circ \tilde{\phi}$ by Φ_n . By (3.3) and (3.4), we have

$$(3.18) \quad \Phi_n \left(\frac{\partial r}{\partial x_i} \right) = \left\{ \prod_{m=1}^{i-1} \Phi_n(x_m x_{m-1} x_m^{-1}) \right\} \cdot \{E - \Phi_n(x_i x_{i-1} x_i^{-1}) + \Phi_n(x_i x_{i-1} x_i^{-1}) \cdot \Phi_n(x_{i+1})\}$$

for $i = 1, 2, \dots, n-1$ and

$$(3.19) \quad \Phi_n \left(\frac{\partial r}{\partial x_n} \right) = \Phi_n(x_1) + \prod_{m=1}^{n-1} \Phi_n(x_m x_{m-1} x_m^{-1}) - E.$$

First we show in the case of $n = 6k + 5$. By combining (3.10), (3.11), (3.12) (3.13) with (3.18), (3.19), we have

$$(3.20) \quad \begin{aligned} \Phi_n \left(\frac{\partial r}{\partial x_1} \right) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - t^{-6k-4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t^{-6k-3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= t^{-6k-4} \begin{pmatrix} t + t^{6k+4} & 1+t \\ 1 & t + t^{6k+4} \end{pmatrix}, \end{aligned}$$

$$(3.21) \quad \begin{aligned} &\Phi_n \left(\frac{\partial r}{\partial x_{3l+2}} \right) \\ &= t^{3l} \left\{ t^{-6k-4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t^{-6k-3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + t^{-6k-2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\} \\ &= t^{3l-6k-4} \begin{pmatrix} t + t^2 & 1+t \\ 1+t^2 & t+t^2 \end{pmatrix} \quad (l = 0, 1, \dots, 2k-1), \end{aligned}$$

$$(3.22) \quad \begin{aligned} &\Phi_n \left(\frac{\partial r}{\partial x_{3l+3}} \right) \\ &= t^{3l} \left\{ t^{-6k-3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - t^{-6k-2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + t^{-6k-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \\ &= t^{3l-6k-3} \begin{pmatrix} 1+t & 1+t^2 \\ t+t^2 & 1+t \end{pmatrix} \quad (l = 0, 1, \dots, 2k-1), \end{aligned}$$

$$(3.23) \quad \begin{aligned} &\Phi_n \left(\frac{\partial r}{\partial x_{3l+4}} \right) \\ &= t^{3l} \left\{ t^{-6k-2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - t^{-6k-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t^{-6k} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \\ &= t^{3l-6k-2} \begin{pmatrix} 1+t^2 & t+t^2 \\ 1+t & 1+t^2 \end{pmatrix} \quad (l = 0, 1, \dots, 2k-1), \end{aligned}$$

$$\begin{aligned}
(3.24) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+2}} \right) &= t^{-4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t^{-3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + t^{-2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
&= t^{-4} \begin{pmatrix} t+t^2 & 1+t \\ 1+t^2 & t+t^2 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(3.25) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+3}} \right) &= t^{-3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - t^{-2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + t^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= t^{-3} \begin{pmatrix} 1+t & 1+t^2 \\ t+t^2 & 1+t+t^2 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(3.26) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+4}} \right) &= t^{-2} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - t^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + t^{-6k-5} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= t^{-6k-5} \begin{pmatrix} t^{6k+3} + t^{6k+4} & 1+t^{6k+4} \\ 1+t^{6k+3} & 1+t^{6k+3} + t^{6k+4} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(3.27) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+5}} \right) &= t^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= t^{-1} \begin{pmatrix} 1+t & 1+t^2 \\ t^2 & 1+t+t^2 \end{pmatrix}.
\end{aligned}$$

By (3.21) and (3.27), we see that

$$\begin{aligned}
(3.28) \quad & \left(\Phi_n \left(\frac{\partial r}{\partial x_2} \right) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+5}} \right) \right) \\
& \sim \begin{pmatrix} t+t^2 & 1+t & 1+t & 1+t^2 \\ 1+t^2 & t+t^2 & t^2 & 1+t+t^2 \end{pmatrix} \\
& \sim \begin{pmatrix} t+t^2 & 0 & 1+t & 1+t^2 \\ 1+t^2 & t & t^2 & 1+t+t^2 \end{pmatrix} \\
& \sim \begin{pmatrix} t+t^2 & 0 & 1+t & 1+t^2 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
& \sim \begin{pmatrix} 0 & 0 & 1+t & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\end{aligned}$$

by the elementary transformations of columns. Then by (3.20)–(3.27) and (3.28), it is easy to see that

$$\left(\Phi_n \left(\frac{\partial r}{\partial x_i} \right) \right)_{i=1,2,\dots,6k+5} \sim \begin{pmatrix} 1+t & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

This implies the desired conclusion.

Next we show in the case of $n = 6k + 7$. By combining (3.14), (3.15), (3.16) (3.17) with (3.18), (3.19), we have

$$\begin{aligned}
(3.29) \quad \Phi_n \left(\frac{\partial r}{\partial x_1} \right) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - t^{-6k-6} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t^{-6k-5} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\
&= t^{-6k-6} \begin{pmatrix} t+t^{6k+6} & 1+t \\ 1 & t+t^{6k+6} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(3.30) \quad & \Phi_n \left(\frac{\partial r}{\partial x_{3l+2}} \right) \\
&= t^{3l} \left\{ t^{-6k-6} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t^{-6k-5} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + t^{-6k-4} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\} \\
&= t^{3l-6k-6} \begin{pmatrix} t+t^2 & 1+t \\ 1+t^2 & t+t^2 \end{pmatrix} \quad (l=0, 1, \dots, 2k-1),
\end{aligned}$$

$$\begin{aligned}
(3.31) \quad & \Phi_n \left(\frac{\partial r}{\partial x_{3l+3}} \right) \\
&= t^{3l} \left\{ t^{-6k-5} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - t^{-6k-4} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + t^{-6k-3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \\
&= t^{3l-6k-5} \begin{pmatrix} 1+t & 1+t^2 \\ t+t^2 & 1+t \end{pmatrix} \quad (l=0, 1, \dots, 2k-1),
\end{aligned}$$

$$\begin{aligned}
(3.32) \quad & \Phi_n \left(\frac{\partial r}{\partial x_{3l+4}} \right) \\
&= t^{3l} \left\{ t^{-6k-4} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - t^{-6k-3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t^{-6k-2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \\
&= t^{3l-6k-4} \begin{pmatrix} 1+t^2 & t+t^2 \\ 1+t & 1+t^2 \end{pmatrix} \quad (l=0, 1, \dots, 2k-1),
\end{aligned}$$

$$\begin{aligned}
(3.33) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+2}} \right) &= t^{-6} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t^{-5} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + t^{-4} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
&= t^{-6} \begin{pmatrix} t+t^2 & 1+t \\ 1+t^2 & t+t^2 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(3.34) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+3}} \right) &= t^{-5} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - t^{-4} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + t^{-3} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= t^{-5} \begin{pmatrix} 1+t & 1+t^2 \\ t+t^2 & 1+t+t^2 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(3.35) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+4}} \right) &= t^{-4} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - t^{-3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + t^{-2} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= t^{-4} \begin{pmatrix} 1+t & t+t^2 \\ 1+t^2 & 1+t+t^2 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(3.36) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+5}} \right) &= t^{-3} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - t^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= t^{-3} \begin{pmatrix} 1+t & 1+t^2 \\ t^2 & 1+t \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(3.37) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+6}} \right) &= t^{-2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - t^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + t^{-6k-7} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= t^{-6k-7} \begin{pmatrix} t^{6k+5} + t^{6k+6} & 1+t^{6k+5} \\ 1 & 1+t^{6k+5} + t^{6k+6} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
(3.38) \quad \Phi_n \left(\frac{\partial r}{\partial x_{6k+7}} \right) &= t^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
&= t^{-1} \begin{pmatrix} 1+t & 1+t^2 \\ t^2 & 1+t+t^2 \end{pmatrix}.
\end{aligned}$$

By (3.30) and (3.38), we see that

$$(3.39) \quad \left(\Phi_n \left(\frac{\partial r}{\partial x_2} \right) \Phi_n \left(\frac{\partial r}{\partial x_{6k+7}} \right) \right) \sim \begin{pmatrix} 0 & 0 & 1+t & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

in the same way as (3.28). Then by (3.29)–(3.38) and (3.39), it is easy to see that

$$A(G_n, \tilde{\rho} \otimes \tilde{\alpha}_n) \sim \begin{pmatrix} 1+t & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

This implies the desired conclusion. \square

By Lemma 3.1 and Theorem 3.6, we succeed to show the nontriviality of Θ_n for $n \equiv 1, 5 \pmod{6}$ by using twisted Alexander ideals.

4. HANDLEBODY-KNOTS

A *handlebody-knot* is a handlebody embedded in S^3 . Two handlebody-knots are said to be *equivalent* if there exists an orientation-preserving self-homeomorphism on S^3 which sends one to the other. A *diagram* of a handlebody-knot H is that of a spatial trivalent graph Γ_H whose regular neighborhood is equivalent to H as a handlebody-knot. A table of genus two handlebody-knots up to six crossings was given in [8]. In this section, we evaluate the twisted Alexander ideals for the handlebody-knots in the table.

For a handlebody-knot H , we denote the fundamental group of the exterior of H in S^3 by G_H . Since the exterior of H and the exterior of Γ_H in S^3 are homeomorphic, it follows that $G_H \cong G_{\Gamma_H}$. Therefore we can obtain a presentation of G_H by taking the Wirtinger presentation for a diagram of H . Moreover, the (twisted) Alexander ideals of H is derived from G_H with (a group representation ρ from G_H to a matrix group G and) an epimorphism α from G_H to an abelian group G_0 as in the case of spatial graphs. Unlike the case of spatial graphs, we cannot specify ρ and α , since there is no canonical meridian system for handlebody-knots. To ensure the invariance, we sum up the (twisted) Alexander ideals over all possible (ρ and) α . Namely, the collection of the (twisted) Alexander ideals of G_H associated with all possible (ρ and) α is an invariant of H . For groups G_1 and G_2 , we denote by $\text{Conj}(G_1, G_2)$ the set of representative elements of conjugacy classes of homomorphisms from G_1 to G_2 , and by $\text{Epi}(G_1, G_2)$ the set of all epimorphisms from G_1 to G_2 . Then we obtain a handlebody-knot invariant of matrix form

$$(E_d(A(G_H, \tilde{\rho} \otimes \tilde{\alpha})))_{\rho \in \text{Conj}(G_H, G), \alpha \in \text{Epi}(G_H, G_0)},$$

where two matrices are assumed to be the same if one can be transformed to the other by permuting rows and columns. Set $G = SL(2; \mathbb{Z}_2)$, $G_0 = \langle t \mid t^2 \rangle$ and $d = 4$. Table 1 lists the invariant of matrix form, where

$$\{(a_{11}, a_{12}, \dots, a_{1n})_{l_1}, (a_{21}, a_{22}, \dots, a_{2n})_{l_2}, \dots, (a_{m1}, a_{m2}, \dots, a_{mn})_{l_m}\}$$

indicates the matrix

$$\begin{pmatrix} \begin{pmatrix} (a_{11}) & (a_{12}) & \dots & (a_{1n}) \\ \vdots & \vdots & & \vdots \\ (a_{m1}) & (a_{m2}) & \dots & (a_{mn}) \end{pmatrix}^{l_1} \\ \vdots \\ \begin{pmatrix} (a_{m1}) & (a_{m2}) & \dots & (a_{mn}) \\ \vdots & \vdots & & \vdots \\ (a_{m1}) & (a_{m2}) & \dots & (a_{mn}) \end{pmatrix}^{l_m} \end{pmatrix}.$$

For example,

$$\{(1, 1, 1)_2, (0, t+1, t+1)_1, (t+1, 0, t+1)_1\} = \begin{pmatrix} (1) & (1) & (1) \\ (1) & (1) & (1) \\ (0) & (t+1) & (t+1) \\ (t+1) & (0) & (t+1) \end{pmatrix}.$$

The second column of Table 1 shows the number of the conjugacy classes of representations of G_H on $SL(2; \mathbb{Z}_2)$ [8]. Then, from the table, we see that the invariant of matrix form works better than the number of the conjugacy classes of representations. Although counting representations is easy way to distinguish two handlebody-knots, the evaluation is not easy if the representation space is big. In such case, the invariant discussed in this section may work well with a small representation space.

H		
0 ₁	11	$\{(1, 1, 1)_{11}\}$
4 ₁	14	$\{(0, 0, 0)_1, (1, 1, 1)_{10}, (0, 1+t, 1+t)_1, (1+t, 0, 1+t)_1, (1+t, 1+t, 0)_1\}$
5 ₁	11	$\{(1, 1, 1)_9, (0, 1+t, 1+t)_1, (1+t, 0, 1+t)_1\}$
5 ₂	14	$\{(0, 0, 0)_1, (1, 1, 1)_{10}, (0, 1+t, 1+t)_1, (1+t, 0, 1+t)_1, (1+t, 1+t, 0)_1\}$
5 ₃	11	$\{(0, 0, 0)_1, (1, 1, 1)_{10}\}$
5 ₄	11	$\{(0, 0, 0)_3, (1, 1, 1)_8\}$
6 ₁	11	$\{(0, 0, 0)_2, (1, 1, 1)_9\}$
6 ₂	11	$\{(0, 0, 0)_1, (1, 1, 1)_{10}\}$
6 ₃	11	$\{(1, 1, 1)_{11}\}$
6 ₄	11	$\{(1, 1, 1)_9, (0, 1+t, 1+t)_1, (1+t, 0, 1+t)_1\}$
6 ₅	11	$\{(1, 1, 1)_{10}, (0, 1+t, 1+t)_1\}$
6 ₆	11	$\{(1, 1, 1)_{10}, (0, 1+t, 1+t)_1\}$
6 ₇	11	$\{(0, 0, 0)_1, (1, 1, 1)_{10}\}$
6 ₈	11	$\{(1, 1, 1)_{11}\}$
6 ₉	14	$\{(0, 0, 0)_3, (1, 1, 1)_{10}, (0, 1+t, 1+t)_1\}$
6 ₁₀	11	$\{(1, 1, 1)_{11}\}$
6 ₁₁	11	$\{(1, 1, 1)_{11}\}$
6 ₁₂	11	$\{(0, 0, 0)_1, (1, 1, 1)_9, (0, 1+t, 1+t)_1\}$
6 ₁₃	14	$\{(0, 0, 0)_1, (1, 1, 1)_{10}, (0, 1+t, 1+t)_1, (1+t, 0, 1+t)_1, (1+t, 1+t, 0)_1\}$
6 ₁₄	17	$\{(0, 0, 0)_9, (1, 1, 1)_8\}$
6 ₁₅	17	$\{(0, 0, 0)_9, (1, 1, 1)_8\}$
6 ₁₆	11	$\{(0, 0, 0)_3, (1, 1, 1)_8\}$

TABLE 1.

5. SURFACE-LINKS

A *surface-link* is a closed surface locally flatly embedded in \mathbb{R}^4 . Two surface-links are said to be *equivalent* if there exists an orientation-preserving self-homeomorphism on \mathbb{R}^4 which sends one to the other. It is well-known that any surface-link can be deformed into a surface-link which has a Morse position with respect to the fourth coordinate and whose maximal and minimal points are in the hyperplane $\mathbb{R}^3 \times \{1\}$ and $\mathbb{R}^3 \times \{-1\}$, respectively, and hyperbolic points are in the hyperplane $\mathbb{R}^3 \times \{0\}$ (cf. [6, 9, 10]). The 0-level cross-section with a “marking” of each vertex is a 4-valent graph which realizes the original surface-link. The diagram is called a *ch-diagram* of the surface-link. In [19], Yoshikawa gave a complete list of surface-links which have ch-diagrams such that the sum of the number of crossings and that of hyperbolic vertices is less than or equal to 10. The *knot group* G_F of a surface-link F , that is the fundamental group of the complement of F , can be calculated by using a Wirtinger presentation of F , refer to [5] for the computation from motion pictures of surface-links and [2] for the computation from ch-diagrams, and the (twisted) Alexander ideals of F is derived from G_F with (a group representation ρ from G_F to a matrix group G and) an epimorphism α from G_H to an abelian group G_0 . Table 2 is Yoshikawa’s original table equipped with the information about the knot groups and the first Alexander ideals associated with the abelianizers. We correct three mistakes: the knot group of $9_1^{1,-2}$, its first Alexander ideal and the first Alexander ideal of $10_2^{0,-2}$. Let α be the homomorphism from G_F to G_0 which sends each Wirtinger generator to t . Then we obtain a surface-link invariant of matrix form

$$(E_d(A(G_F, \tilde{\rho} \otimes \tilde{\alpha})))_{\rho \in \text{Conj}(G_H, G), d=1,2,\dots},$$

where two matrices are assumed to be the same if one can be transformed to the other by permuting rows. Set $G = SL(2; \mathbb{Z}_2)$ and $G_0 = \langle t \mid t^2 \rangle$. Table 3 lists this invariant for the surface-links in Yoshikawa’s table, where we omit the last consecutive 1’s in each pair of parentheses, that is,

$$\{(a_{11}, a_{12}, \dots, a_{1n_1-1}, 1)_{l_1}, \dots, (a_{m1}, a_{m2}, \dots, a_{mn_m-1}, 1)_{l_m}\}$$

represents

$$\{(a_{11}, a_{12}, \dots, a_{1n_1-1}, 1, 1, 1, \dots)_{l_1}, \dots, (a_{m1}, a_{m2}, \dots, a_{mn_m-1}, 1, 1, 1, \dots)_{l_m}\}$$

and indicates the matrix with infinite columns as shown in §4.

Consider the cases of $8_1^{-1,-1}$ and $9_1^{1,-2}$. They have the same first Alexander ideals. However they are distinguished by comparing the second twisted Alexander ideals of $8_1^{-1,-1}$ for all the group representations to $SL(2; \mathbb{Z}_2)$ and that of $9_1^{1,-2}$ for a group representation to $SL(2; \mathbb{Z}_2)$. Thus $8_1^{-1,-1}$ and $9_1^{1,-2}$ gives an example of two surface-links which are not distinguished by the first Alexander ideals but distinguished by the twisted Alexander ideals. Note that they are also distinguished by the 0th Alexander ideals or the number of group representations to $SL(2; \mathbb{Z}_2)$, and moreover, the surface-links are clearly different since $8_1^{-1,-1}$ does not have an orientable component but $9_1^{1,-2}$ does.

Remark 5.1. When we have an example of two classical-knots which are not distinguished by the Alexander ideals but distinguished by twisted Alexander ideals, we can easily construct an example of surface-links which satisfies the same property by taking the Artin’s spinning process [1] of the classical-knots. It is known that

the Kinoshita-Terasaka knot K_{KT} and the Conway knot K_C are not distinguished by the Alexander ideals but distinguished by twisted Alexander ideals with the parabolic representations to $SL(2; \mathbb{Z}_7)$, see [18]. Thus the spun K_{KT} and the spun K_C gives an example of two surface-links which are not distinguished by the Alexander ideals but distinguished by twisted Alexander ideals.

F	$\pi(\mathbb{R}^4 - F)$	ideal	polynomial
0_1	\mathbb{Z}	(1)	1
2_1^+	\mathbb{Z}	(1)	1
2_1^{-1}	\mathbb{Z}_2	(1)	1
$6_1^{0,1}$	$\mathbb{Z} \oplus \mathbb{Z}$	$(x-1, y-1)$	1
$7_1^{0,-2}$	$\langle x, y \mid yxyx^{-1} \rangle$	$(x+1, y-1)$	1
8_1	$\langle x_1, x_2 \mid x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1} \rangle$	$(x^2 - x + 1)$	$x^2 - x + 1$
$8_1^{1,1}$	$\mathbb{Z} \oplus \mathbb{Z}$	$(x-1, y-1)$	1
$8_1^{-1,-1}$	$\langle x, y \mid xyxy^{-1}, x^{-2}y^2 \rangle$	$(x+1, y+1, 2)$	1
9_1	$\langle x_1, x_2 \mid x_1x_2^{-1}x_1x_2x_1^{-1}x_2^{-1} \rangle$	$(x-2)$	$x-2$
$9_1^{0,1}$	$\langle x, y \mid x^{-1}y^{-1}xyx^{-1}yxy^{-1} \rangle$	$((x-1)(y-1), (y-1)^2)$	$y-1$
$9_1^{1,-2}$	$\langle x, y \mid xyxy^{-1}, x^2 \rangle$	$(x+1, y+1, 2)$	1
10_1	$\langle x_1, x_2 \mid x_1^{-1}x_2x_1x_2^{-1}x_1x_2x_1^{-1}x_2^{-1}x_1x_2^{-1} \rangle$	$(x^2 - 3x + 1)$	$x^2 - 3x + 1$
10_2	$\langle x_1, x_2 \mid x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1}, x_1^2x_2x_1^{-2}x_2^{-1} \rangle$	$(x+1, 3)$	1
10_3	$\langle x_1, x_2 \mid x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1}, x_1^3x_2x_1^{-3}x_2^{-1} \rangle$	$(x^2 + x + 1, 2)$	1
10_1^1	$\langle x_1, x_2 \mid x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1} \rangle$	$(x^2 - x + 1)$	$x^2 - x + 1$
$10_1^{0,1}$	$\langle x, y \mid x^{-1}y^{-1}x^{-1}yxyxy^{-1} \rangle$	$((x-1)(xy+1), (y-1)(xy+1))$	$xy+1$
$10_2^{0,1}$	$\langle x, y \mid x^2yx^{-2}y^{-1} \rangle$	$((x-1)(x+1), (y-1)(x+1))$	$x+1$
$10_1^{1,1}$	$\mathbb{Z} \oplus \mathbb{Z}$	$(x-1, y-1)$	1
$10_1^{0,0,1}$	$\langle x, y, z \mid y^{-1}x^{-1}zxyzy^{-1} \rangle$	(0)	0
$10_1^{0,-2}$	$\langle x, y \mid x^{-1}y^{-1}xyx^{-1}yxy \rangle$	$(2x+y-1, 4)$	1
$10_2^{0,-2}$	$\langle x, y \mid xy^2x^{-1}y^{-2}, yx^{-1}y^{-1}xyx^{-1}yx \rangle$	$(2x+y-1, 4, 2x^2+2)$	1
$10_1^{-1,-1}$	$\langle x, y \mid x^2y^2, yxyxyx^{-1}y^{-1}x^{-1} \rangle$	$(x+1, y+1, 4)$	1
$10_1^{-2,-2}$	$\langle x, y \mid xyxy^{-1}, x^{-2}y^2 \rangle$	$(x+1, y+1, 2)$	1

TABLE 2.

ACKNOWLEDGMENT

The authors are grateful to Professor Charles Livingston and Professor Kokoro Tanaka for their valuable comments.

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F	invariant
0_1	$\{(0, 1)_3\}$
2_1^1	$\{(0, 1)_3\}$
2_1^{-1}	$\{(1)_1, (1+t, 1)_1\}$
$6_1^{0,1}$	$\{(0, 1)_4, (0, 0, 1)_1, (0, 1+t, 1)_2, (0, 0, 1+t, 1)_1\}$
$7_1^{0,-2}$	$\{(0, 1)_2, (0, 0, 1)_1, (0, 1+t, 1)_2, (0, 0, 1+t, 1)_1\}$
8_1	$\{(0, 1)_2, (0, 0, 1)_1, (0, 0, 1+t, 1)_1\}$
$8_1^{1,1}$	$\{(0, 1)_4, (0, 0, 1)_1, (0, 1+t, 1)_2, (0, 0, 1+t, 1)_1\}$
$8_1^{-1,-1}$	$\{(0, 1)_3, (0, 0, 1+t, 1)_1\}$
9_1	$\{(0, 1)_4\}$
$9_1^{0,1}$	$\{(0, 1)_4, (0, 0, 0, 1)_2, (0, 0, 1+t, 1)_3\}$
$9_1^{1,-2}$	$\{(0, 1)_3, (0, 1+t, 1)_1, (0, 0, 1+t, 1)_1\}$
10_1	$\{(0, 1)_2, (0, 0, 1+t, 1)_1\}$
10_2	$\{(0, 1)_4\}$
10_3	$\{(0, 1)_4\}$
10_1^1	$\{(0, 1)_2, (0, 0, 1)_1, (0, 0, 1+t, 1)_1\}$
$10_1^{0,1}$	$\{(0, 1)_3, (0, 0, 1)_2, (0, 0, 0, 1)_3, (0, 0, 1+t, 1)_2\}$
$10_2^{0,1}$	$\{(0, 1)_3, (0, 0, 1)_2, (0, 0, 0, 1)_2, (0, 0, 1+t, 1)_3\}$
$10_1^{1,1}$	$\{(0, 1)_4, (0, 0, 1)_1, (0, 1+t, 1)_2, (0, 0, 1+t, 1)_1\}$
$10_1^{0,0,1}$	$\{(0, 0, 0, 1)_{16}, (0, 0, 0, 0, 1)_4, (0, 0, 0, 1+t, 1)_8, (0, 0, 0, 0, 1+t, 1)_3\}$
$10_1^{0,-2}$	$\{(0, 0, 1)_2, (0, 0, 1+t, 1)_3\}$
$10_2^{0,-2}$	$\{(0, 0, 1)_2, (0, 0, 1+t, 1)_3\}$
$10_1^{-1,-1}$	$\{(0, 1)_1, (0, 1+t, 1)_2, (0, 0, 1+t, 1)_1\}$
$10_1^{-2,-2}$	$\{(0, 1)_3, (0, 0, 1+t, 1)_1\}$

TABLE 3.

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